

Stationary boundaries on the space of amenable subgroups

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The space $\text{Sub}(G)$ of subgroups of a discrete countable group G is naturally identified with a closed subset of $\{0, 1\}^G$ endowed with the product topology. The induced topology, known as the *Chabauty topology*, makes $\text{Sub}(G)$ a compact metrizable space. The conjugation action of G on its subgroups induces an action by homeomorphisms $G \curvearrowright \text{Sub}(G)$. The set $\text{Sub}_{\text{am}}(G)$ of amenable subgroups of G is closed in $\text{Sub}(G)$, and a reason to study the dynamics of $G \curvearrowright \text{Sub}_{\text{am}}(G)$ is its connection with the C^* -simplicity of G .

Recall that a measurable action of G on a probability space (X, η) is μ -stationary if $\eta = \sum_{g \in G} \mu(g)g_*\eta$. A μ -stationary probability measure on $\text{Sub}_{\text{am}}(G)$ is called an *amenable μ -stationary random subgroup* (or *amenable μ -SRS*). This notion turns out to be useful to characterize C^* -simplicity of countable groups by work of Hartman-Kalantar [3], who proved that a countable group G is C^* -simple if and only if there exists a non-degenerate probability measure $\mu \in \text{Prob}(G)$ such that $\text{Sub}_{\text{am}}(G)$ admits a unique μ -stationary probability measure (which must be $\delta_{\{e_G\}}$). We address the existence of probability measures with the opposite property.

Question 1. *Let G be a countable group. Does there exist a non-degenerate probability measure μ on G that admits an amenable μ -SRS distinct from $\delta_{\{e_G\}}$?*

It follows from [3] that Question 1 is of interest only when G is C^* -simple. Our results deal with the following class of amenable SRSs.

Definition 2. Let μ be a probability measure on a countable group G and denote by \mathbb{P}_μ the law of the right μ -random walk on G . A μ -stationary probability measure η on the space $\text{Sub}_{\text{am}}(G)$ is called an *amenable boundary μ -stationary random subgroup* if, for \mathbb{P}_μ -almost every trajectory $(w_n)_{n \geq 0} \in G^{\mathbb{N}}$, the sequence $(w_n * \eta)_{n \geq 0}$ converges in the weak*-topology to a Dirac mass in $\text{Sub}_{\text{am}}(G)$.

Our first main theorem provides a condition that guarantees the existence of such a measure.

Theorem 3. *Let G be a countable group. Suppose that there exists a non-trivial subgroup H such that for all finite subsets $Q, Z \subseteq G$ there is $b \in G$ such that*

$$bZ, bZ^{-1} \subseteq \{g \in G : Q \cap H = Q \cap gHg^{-1}\}.$$

Then there exists a non-degenerate, symmetric and finite-entropy probability measure μ on G such that $\overline{\text{Orb}_G(H)}$ supports a μ -boundary SRS distinct from $\delta_{\{e_G\}}$.

In particular, if H is amenable then G admits an amenable boundary μ -SRS distinct from $\delta_{\{e_G\}}$.

The condition appearing in the statement can be viewed as a strong form of recurrence for the action of G on the orbit of H . The strategy we use to construct μ is the method of record times used by Frisch-Hartman-Tamuz-Vahidi Ferdowsi [2]

to prove that any non-hyper-FC-central group admits non-degenerate probability measures with a non-trivial Poisson boundary.

The criterion in Theorem 3 applies to (permutational) wreath products, which include groups that are C^* -simple, and to Thompson's group F , whose C^* -simplicity is equivalent to its non-amenability [4], and therefore remains an open problem.

Corollary 4. *There exists a non-degenerate, symmetric and finite-entropy probability measure μ on Thompson's group F such that $\text{Sub}_{\text{ab}}(F) = \{H \in \text{Sub}(F) : H \text{ is abelian}\}$ supports a non-atomic μ -boundary.*

Following [1], we say that a subgroup H of G is called *normalish* if $\bigcap_{z \in Z} zHz^{-1}$ is infinite for every finite subset $Z \subseteq G$. Our second main theorem connects this concept to Question 1.

Theorem 5. *Let μ be a non-degenerate probability measure on a countable group G and let η be a μ -stationary probability measure on $\text{Sub}(G)$. Suppose that $(\text{Sub}(G), \eta)$ is a μ -boundary of G , and that η is not a Dirac mass on a finite normal subgroup of G . Then η is supported on normalish subgroups of G .*

As a consequence, we obtain the next result.

Corollary 6. *Let G be a countable group with no finite normal subgroups and no amenable normalish subgroups. Then, for every non-degenerate $\mu \in \text{Prob}(G)$, the action of G on the Poisson boundary of (G, μ) is essentially free.*

Finally, we observe that one cannot expect a converse to Theorem 5. More precisely, we show that many Baumslag-Solitar groups have amenable normalish subgroups but their space of amenable subgroups is countable. Thus, they admit no non-trivial amenable μ -SRSs for any non-degenerate μ . This is the case in particular for $\text{BS}(2, 3) = \langle a, t \mid ta^2t^{-1} = a^3 \rangle$.

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