Prof. Dr. Dr. Katrin Tent
Marco Amelio
Anna Cascioli
Raquel Murat García

Summer Semester 2025
Universität Münster

Seminar on the Boone-Higman Conjecture and Twisted Brin-Thompson Groups

The seminar will take place on Tuesdays from 10:15 to 12:00.

Summary of the topic and goals of the seminar

The celebrated Higman Embedding Theorem states that a finitely generated group admits a computable presentation if and only if it embeds into a finitely presented group. As a natural companion to this result, Boone and Higman conjectured in the 1970s that a finitely generated group has a solvable word problem if and only if it embeds into a simple finitely presented group. This conjecture has been proved for many interesting classes of groups, but the general case remains open. One of the main challenges lies in finding suitable target groups for embeddings, specifically, identifying (infinite) finitely presented simple groups into which groups with a solvable word problem can embed. A natural family of such groups includes the twisted Brin-Thompson groups SV_G, associated with the action of a group G on a set S. Under certain conditions on the action of G on S, SV_G is a finitely presented simple group, and G embeds into it.

This seminar will consist of three main parts:

- Group theoretic basics (talks 1 to 4): we will cover the group theoretic and algorithmic concepts and results necessary to understand and motivate the BH conjecture. Topics include (but are not limited to) free groups and group presentations, group actions, HNN-extesnions, Cayley graphs and hyperbolic groups.

- The twisted Brin-Thompson groups (talks 5 to 7): we will introduce the Cantor set, Thompson's group V (viewed as a subgroup of the group of homeomorphisms of the Cantor set), and its generalizations, the twisted Brin-Thompson groups SV_G. We will also (partially) prove results related to embeddings, finite presentability and simplicity for SV_G.

- Hyperbolic groups satisfy the BH-conjecture: we will introduce Dehn presentations, prove that hyperbolic groups have solvable word problem, and then study the proof of Belk, Bleak, Matucci and Zaremsky that hyperbolic groups satisfy the Boone-Higman conjecture.

List of talks

We provide now the list of talks (with the scheduled date, the main topics to be covered and the recommended literature). All the talks must be given in English and should last for 1 hour 30 minutes.

Talk 1: Group theoretic basics I (Sophia)

Definition of free groups via (reduced) words over an alphabet, universal property of free groups, free groups are determined by the cardinality of their basis. Introduce groups given by generators and relations, finitely generated and finitely presented groups. Group actions, faithful, free, transitive and oligomorphic actions. Cayley's theorem.

Date: 15th of April

Literature: [11, Sections 2.2 and 2.3], [9, Sections 7.1 to 7.4], [6, Sections 1.4 and 1.6]. Notes of Martin Bays, Sections 1.1, 3.0 and 4.0.

Talk 2: Group theoretic basics II (Elias)

Definition of a graph (in the sense of Serre), group actions on graphs, Cayley graphs. Free products: definition, normal forms, universal property. HNN-extensions: definition, normal forms, Britton's Lemma, embedding of the vertex group. Definition of a hyperbolic metric space and a hyperbolic group, relevant properties of hyperbolic groups.

Date: 22nd of April

Literature: [11, Sections 7.3 and 7.5], [6, Sections 2.1, 2.10 and 2.14]. Notes of Martin Bays, Sections 2.1, 2.3 and 5.1.

Talk 3: Algorithmic properties and the word problem (Julius)

Turing machines: definition and examples. Recursively enumerable and enumerable sets: definition. Existence of non-recursively enumerable sets of $\mathbb{N}$. Definition of recursively presented group. The word problem: statement, independece on the finite generating set, easy examples of groups with solvable word problem.

Date: 29th of April

Literature: [13, Pages 420 to 425] and [12, Section 5.2]

Talk 4: Introduction to the Boone-Higman conjecture (Theodor)

Computably presented groups: recall definition, proof that any finitely generated subgroup of a finitely presented group is computably presented. The Higman embedding theorem: statement and proof of the 'easy' implication. The Boone-Higman conjecture: motivation. Proof that finitely presented simple groups have solvable word problem. Partial results.

Date: 6th of May

Literature: [4, Sections 1 to 3]

Talk 5: Twisted Brin-Thompson groups I

The Cantor set: definition, homeomorphisms. Thompson's group V: definition as a subgroup of homeomorphisms of the Cantor set. Criterion for the simplicity of a subgroup of homeomorphisms. Higher dimensional Thompson groups: definition of the Brin-Thompson groups. Twisted Brin-Thompson groups: definition and embedding property.

Date: 13th of May

Literature: [8, Section 6], [1, Sections 2 and 3], [5] and [2, Section 1]

Talk 6: Twisted Brin-Thompson groups II (Lars)

Beginning of the proof of simplicity of Twisted Brin-Thompson groups: Twisted Brin-Thompson grupoids.

Date: 20th of May

Literature: [2, Sections 1 and 2]

Talk 7: Twisted Brin-Thompson groups III

End of the proof of simplicity of Twisted Brin-Thompson groups: generators for the Brin-Thompson and Twisted Brin-Thompson groups and proof of their simplicity with a criterion for being perfect.

Date: 20th of May

Literature: [10, Theorem 25], [2, Section 3] and [7].

Talk 8: Hyperbolic groups satisfy the BH-conjecture I (Jonas)

Dehn presentations, proof that a group with a Dehn presentation have solvable word problem. Recall the definition of a hyperbolic group and proof that they admit a Dehn presentation.

Date: 27th of May

Literature: [11, Chapter 7.4]

Talk 9: Hyperbolic groups satisfy the BH-conjecture II

Date: 3rd of June

Literature: [3]

Talk 10: Hyperbolic groups satisfy the BH-conjecture III

Date: 17th of June

Literature: [3]

Talk 11: Hyperbolic groups satisfy the BH-conjecture IV

Date: 24th of June
Literature: [3]

Talk 12: Hyperbolic groups satisfy the BH-conjecture V

Date: 1st of July
Literature: [3]

References

[1] James Belk and Matthew CB Zaremsky. "Twisted Brin–Thompson groups". In: *Geometry & Topology* 26.3 (2022), pp. 1189–1223.

[2] James Belk and Matthew CB Zaremsky. "Twisted Brin–Thompson groups". In: *Geometry & Topology* 26.3 (2022), pp. 1189–1223.

[3] James Belk et al. "Hyperbolic groups satisfy the Boone-Higman conjecture". In: *arXiv preprint arXiv:2309.06224* (2023).

[4] James Belk et al. "Progress around the Boone-Higman conjecture". In: *arXiv preprint arXiv:2306.16356* (2023).

[5] Collin Bleak, Luna Elliott, and James Hyde. "Sufficient conditions for a group of homeomorphisms of the Cantor set to be two-generated". In: *Journal of the Institute of Mathematics of Jussieu* 23.6 (2024), pp. 2825–2858.

[6] Oleg Vladimirovič Bogopolskij. *Introduction to group theory*. Vol. 6. European Mathematical Society, 2008.

[7] Matthew G Brin. "On the baker's map and the simplicity of the higher dimensional Thompson groups nV". In: (2010).

[8] James W Cannon. "Introductory notes on Richard Thompson's groups". In: *Enseign. Math.(2)* 42.3-4 (1996), p. 215.

[9] Cornelia Druţu and Michael Kapovich. *Geometric group theory*. Vol. 63. American Mathematical Soc., 2018.

[10] Johanna Hennig and Francesco Matucci. "Presentations for the higher-dimensional Thompson groups nV". In: *Pacific Journal of Mathematics* 257.1 (2012), pp. 53–74.

[11] Clara Löh. *Geometric group theory*. Springer, 2017.

[12] John Meier. "Groups, graphs and trees". In: *(No Title)* (2008).

[13] Joseph J Rotman. *An introduction to the theory of groups*. Vol. 148. Springer Science & Business Media, 2012.